Optimization Algorithm Design via Electric Circuits

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Distributed convex optimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\intercal) \end{array}$$

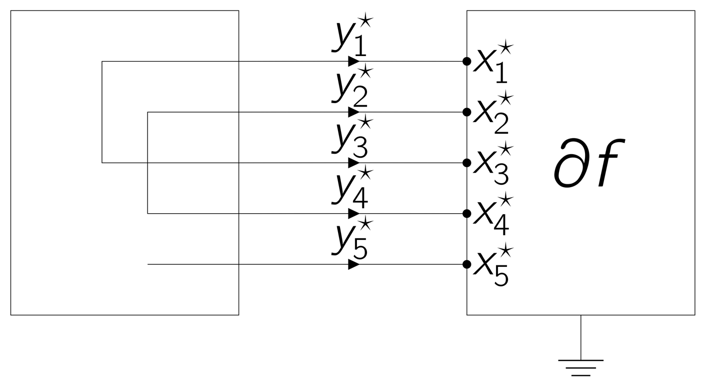
- $f : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ is closed, convex, and proper
- n nets $N_1, \dots, N_n$ forming a partition of $\{1, \dots, m\}$
- $E \in \mathbf{R}^{n \times m}$ is a selection matrix

$$E_{ij} = \begin{cases} +1 & \text{if } j \in N_i \\ 0 & \text{otherwise} \end{cases}$$

Circuit interpretation: KKT conditions

$y \in \partial f(x)$ (resistor)
$x \in \mathcal{R}(E^\intercal)$ (KVL)
$y \in \mathcal{N}(E)$ (KCL)

Static interconnect



Circuit interpretation: Dynamic interconnect

$y(t) \in \partial f(x(t))$ (resistor)

$v(t) = A^\intercal \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ (KVL)

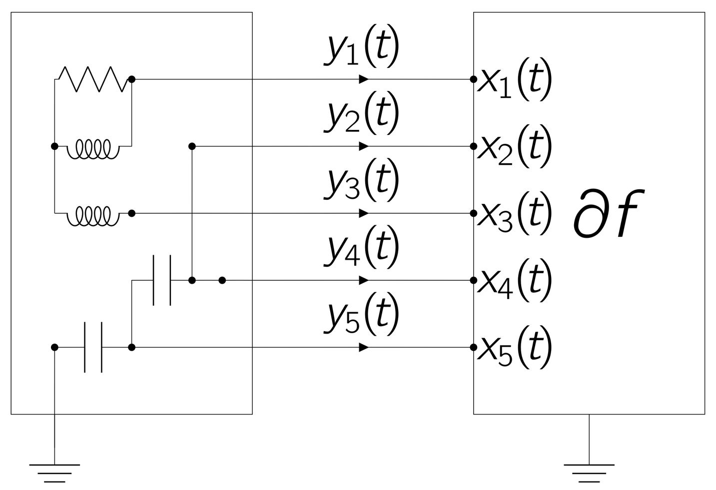
$Ai(t) = \begin{bmatrix} -y(t) \\ 0 \end{bmatrix}$ (KCL)

$v_\mathcal{R}(t) = D_\mathcal{R} i_\mathcal{R}(t)$ (R)

$v_\mathcal{L}(t) = D_\mathcal{L} \frac{d}{dt} i_\mathcal{L}(t)$ (L)

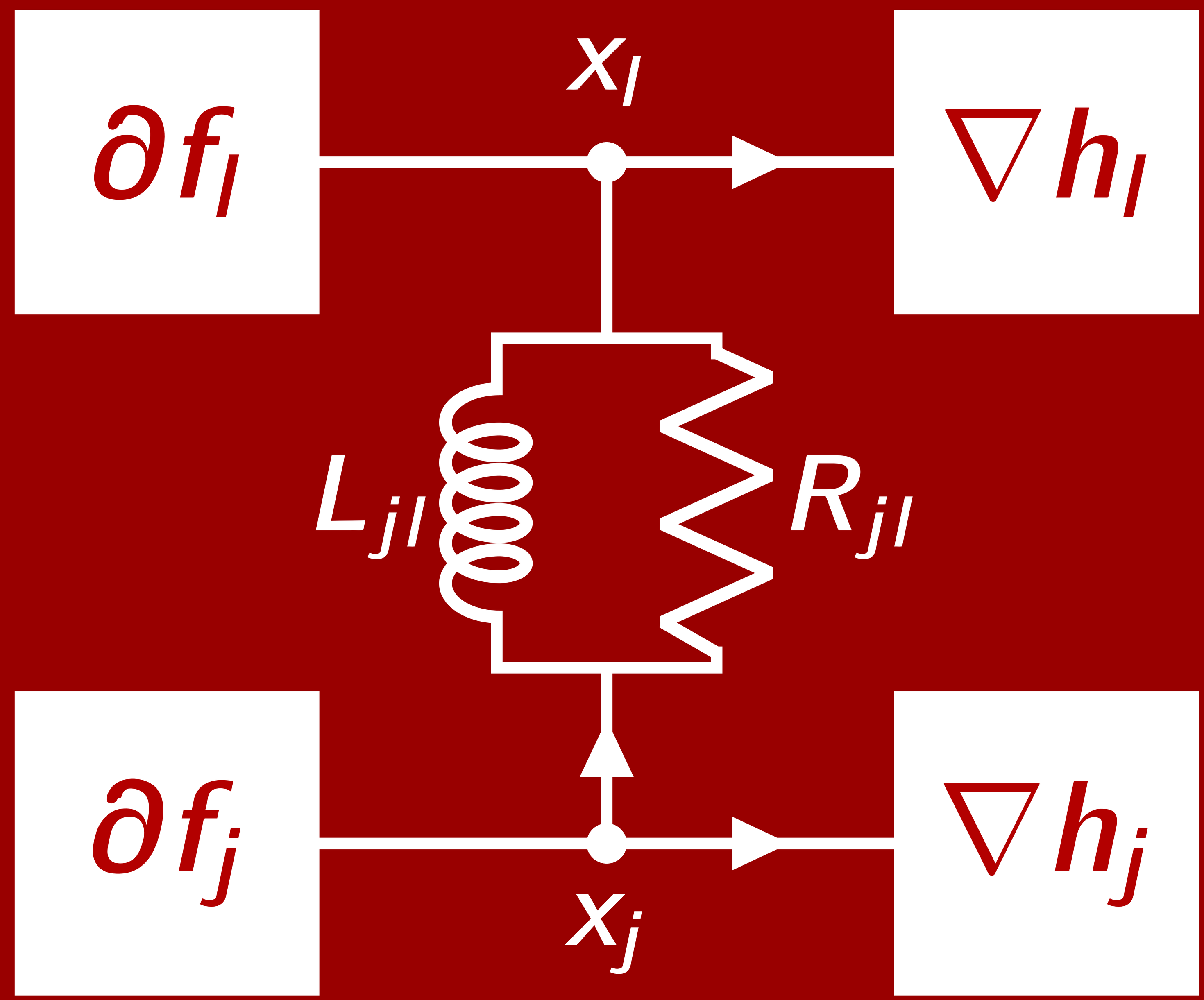
$i_\mathcal{C}(t) = D_\mathcal{C} \frac{d}{dt} v_\mathcal{C}(t)$ (C)

Dynamic interconnect



Continuous-time convergence

- energy dissipation leads to convergence
  - $\mathcal{E}(t) = \frac{1}{2}\|v_\mathcal{C}(t) - v_\mathcal{C}^\star\|^2_{D_\mathcal{C}} + \frac{1}{2}\|i_\mathcal{L}(t) - i_\mathcal{L}^\star\|^2_{D_\mathcal{L}}$
  - $\frac{d}{dt}\mathcal{E} \le -\langle x(t) - x^\star, y(t) - y^\star \rangle \le 0$
  - $\lim_{t\to\infty} x(t) = x^\star$
- not every discretization leads to a convergent algorithm



This is a convergent optimization algorithm!

PG-EXTRA

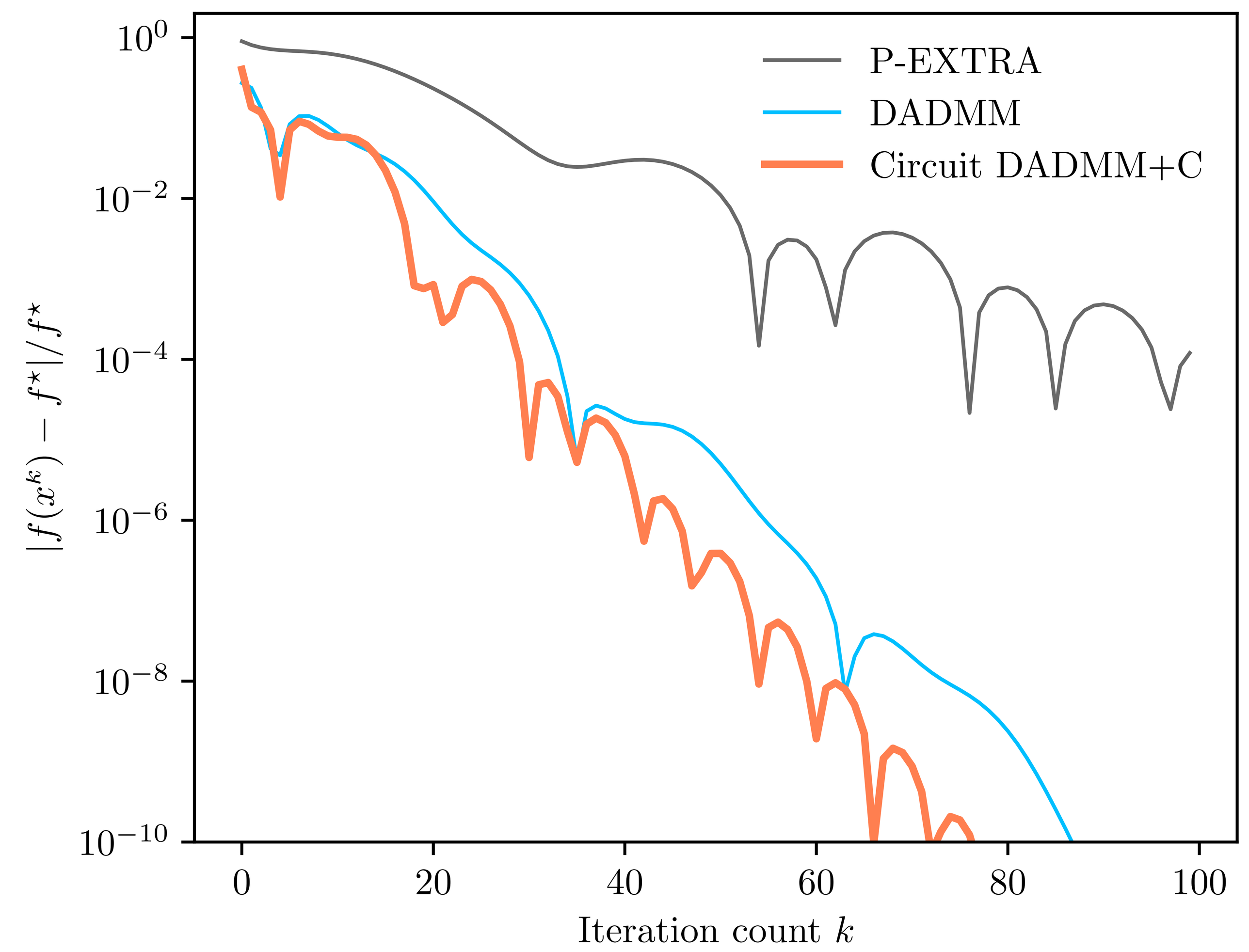
Automatic discretization

- find discretization preserving the proof structure
  - $\mathcal{E}_k = \frac{1}{2}\|v_\mathcal{C}^k - v_\mathcal{C}^\star\|^2_{D_\mathcal{C}} + \frac{1}{2}\|i_\mathcal{L}^k - i_\mathcal{L}^\star\|^2_{D_\mathcal{L}}$
  - $\mathcal{E}_{k+1} - \mathcal{E}_k + \eta\langle x^k - x^\star, y^k - y^\star \rangle \le 0$ for some $\eta > 0$
  - $\lim_{k\to\infty} x^k = x^\star$
- automate using computer-assisted proof framework PEP

Design your algorithm via circuits!

step 1: create the static interconnect representing the optimality conditions of your problem
step 2: design your algorithm: design RLC circuit that relaxes to the static interconnect in equilibrium
step 3: write the V-I relations: this is a convergent dynamics by the construction
step 4: leverage our PEP-based automatic discretization package `ciropt` and obtain discrete algorithm
step 5: your algorithm is ready to use!

Numerical result: DADMM+C



Contribution

- easy-to-use framework for designing new convergent optimization algorithms via RLC circuits
- identified electric circuits for many standard methods
- established convergence proof structure
- PEP-based automated discretization
  - preserves proof structure
  - open-source package `ciropt`